

# What You Can See in Limited Data Tomography

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<http://equinto.math.tufts.edu>  
(jokes, too)

DTU High School Day, November 17, 2016!

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. . .We'll see! . . .

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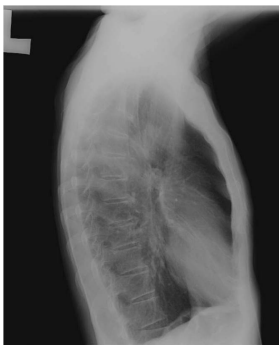
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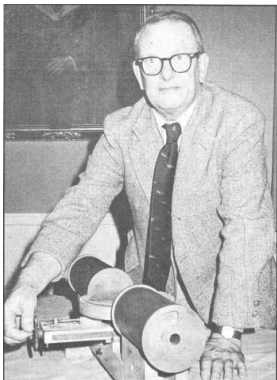
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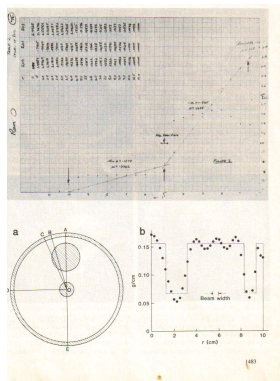
Allan won the 1979 Nobel Prize in Medicine! (early AM)...taught!

# Allan Cormack + his CT Scanner

Allan + Scanner



His calculations from 1963



# Complete Tomographic Data

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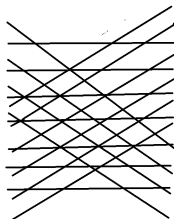
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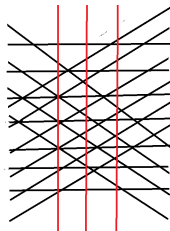
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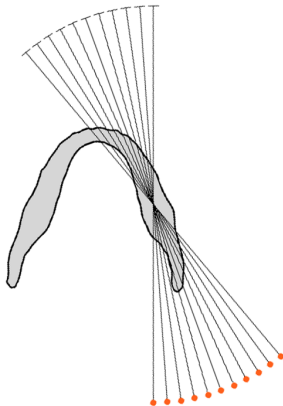


Vertical-ish lines are missing from the data set.



# Limited Angle CT in Dental Imaging

Dental Scanner—head goes in “Π” Jaw showing X-ray projection angles



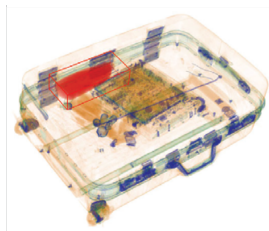
<http://www.siltanen-research.net>

# Limited Angle CT in Luggage Testing

Luggage Scanner



Sample Luggage scan



Scanner moves above and below suitcase →

Analogic COBRA carry-on luggage scanner

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**Now do math then analyze limited angle CT reconstructions!**

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My Answer: The bones *and* their edges (boundaries)!



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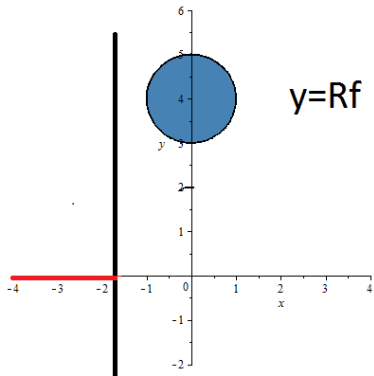


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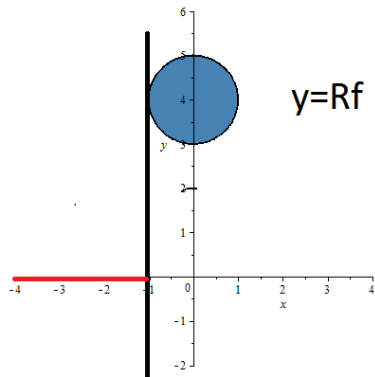
**Answer:** The beams tangent to the edges (boundaries) of the bones!

*Now see why mathematically.*

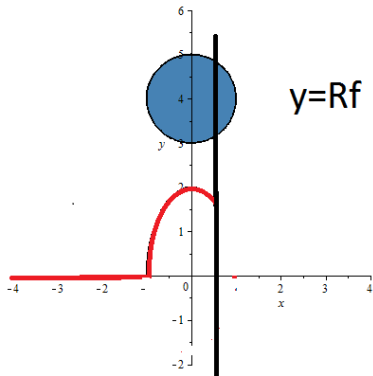
# Profile curve of a disk of radius one from the south (vertical lines)



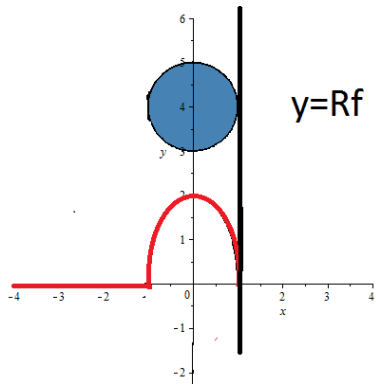
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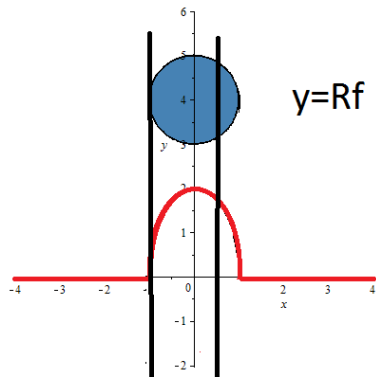
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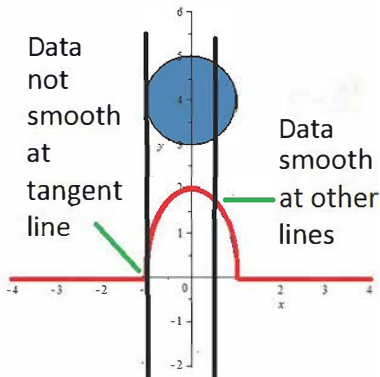
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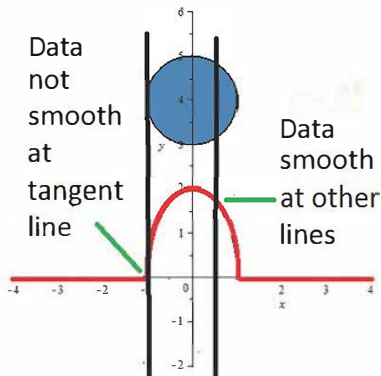


## Profile curve of a disk of radius one from the south (vertical lines)



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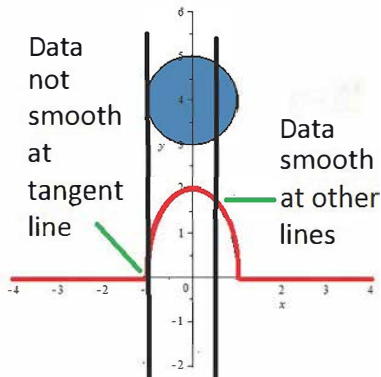
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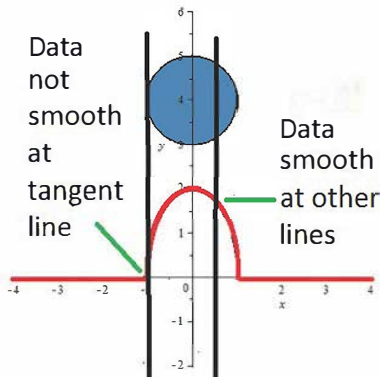
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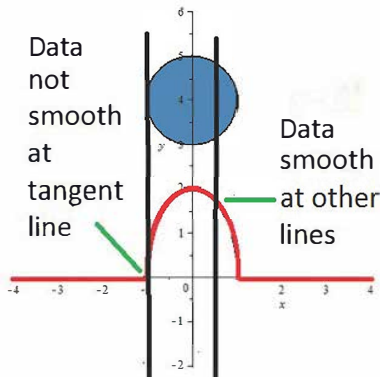
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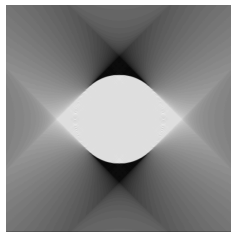
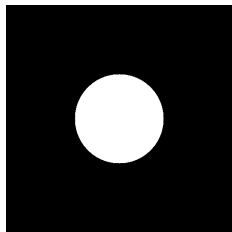
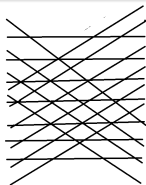


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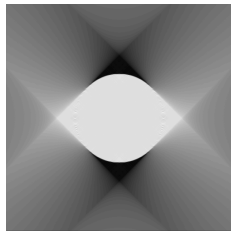
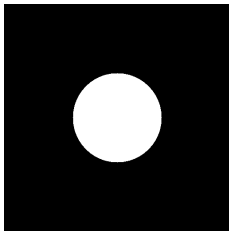
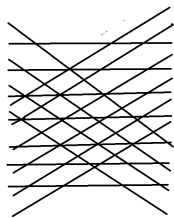
## Example

Limited angle CT data of a disk over lines with slope between  $-1$  and  $1$ .



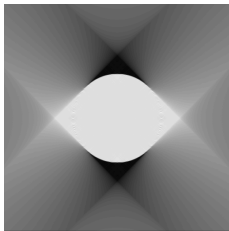
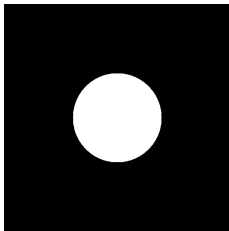
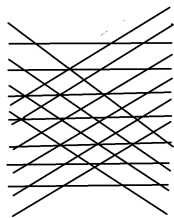
[Frikel, Q 2013] *Left: disk, Right: FBP reconstruction*

**The data set:** all lines with slope between  $-1$  and  $+1$ .



**Which boundaries of the disk are visible in the reconstruction?**

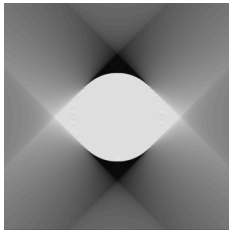
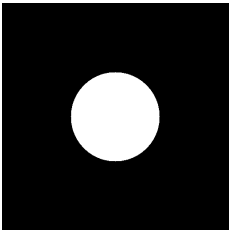
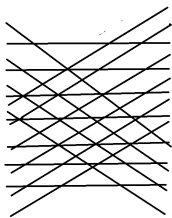
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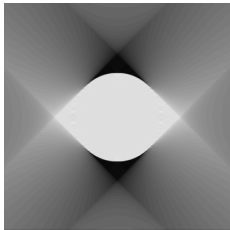
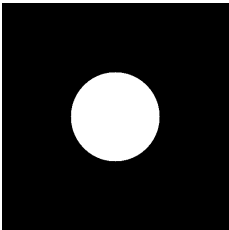
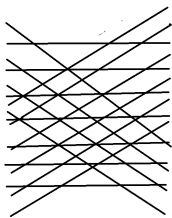
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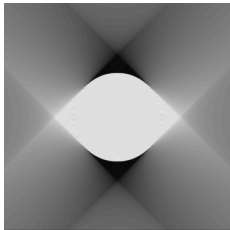
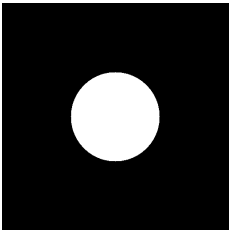
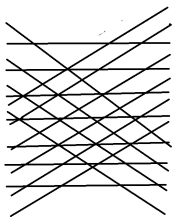
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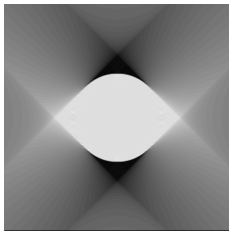
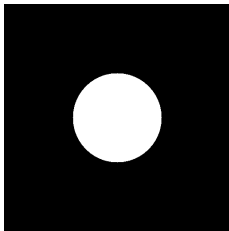
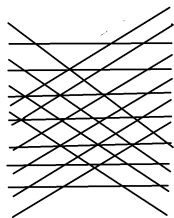
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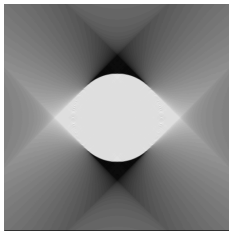
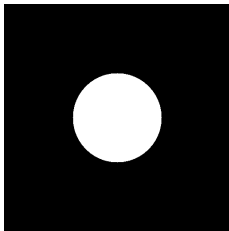
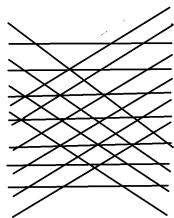
This illustrates the moral: **a part of the boundary of an object will be visible in the reconstruction if it is tangent to a line in the data set!**

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**Which boundaries of the disk are not visible in the reconstruction?**

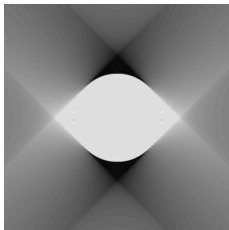
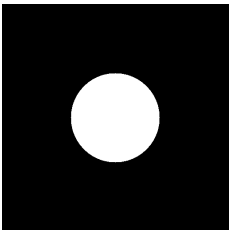
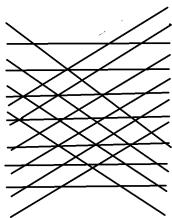
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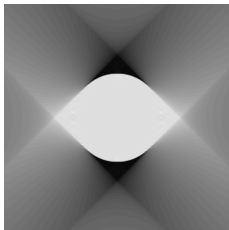
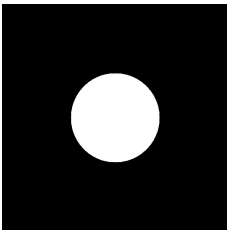
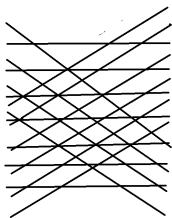
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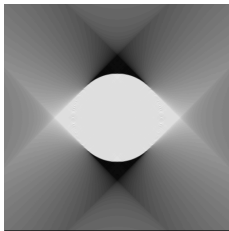
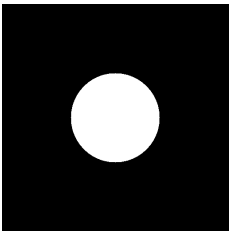
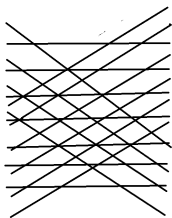
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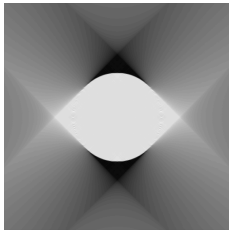
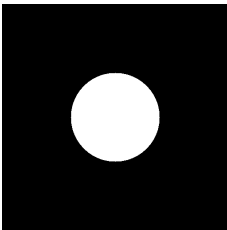
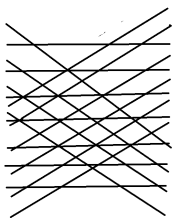
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We learned that, if no line in the data set is tangent to a boundary, that boundary will be hard to see in the reconstruction.

**Is that true in this picture? YES!**

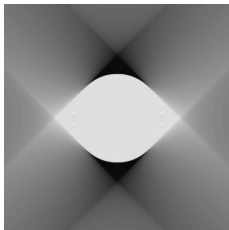
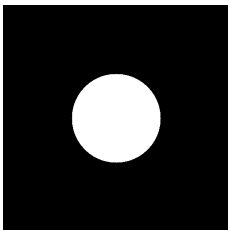
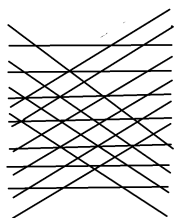
This illustrates the moral: **a part of the boundary of an object will be hard to see in the reconstruction if *no* line in the data set is tangent to it!**

**The data set:** all lines with slope between  $-1$  and  $+1$ .



**How do the streaks relate to the data set and object?**

**The data set:** all lines with slope between  $-1$  and  $+1$ .

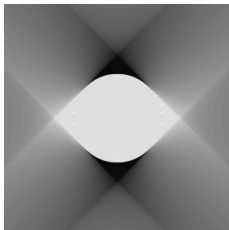
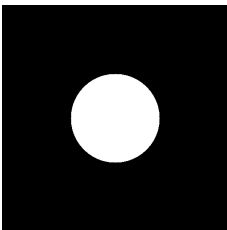
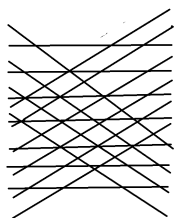


**How do the streaks relate to the data set and object?**

What are their slopes?



**The data set:** all lines with slope between  $-1$  and  $+1$ .

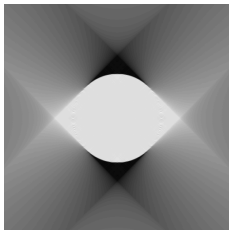
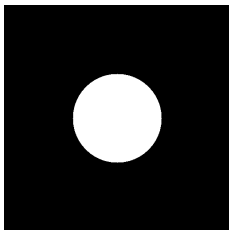
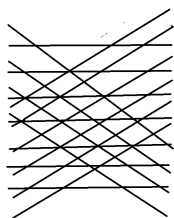


**How do the streaks relate to the data set and object?**

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Either  $-1$  or  $+1$ —they are lines at the ends of the data set.

**The data set:** all lines with slope between  $-1$  and  $+1$ .



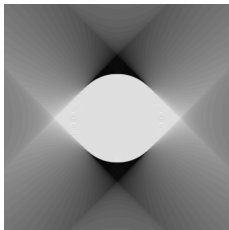
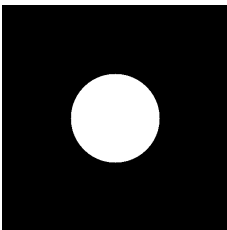
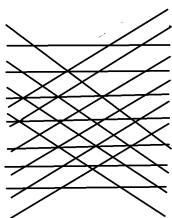
**How do the streaks relate to the data set and object?**

What are their slopes?

Either  $-1$  or  $+1$ —they are lines at the ends of the data set.

**How do the streak lines relate to the object?**

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**How do the streaks relate to the data set and object?**

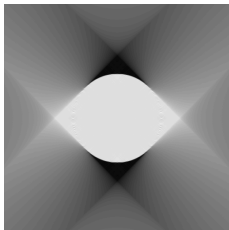
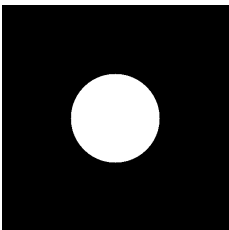
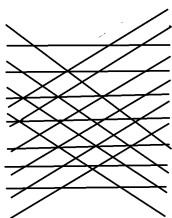
What are their slopes?

Either  $-1$  or  $+1$ —they are lines at the ends of the data set.

**How do the streak lines relate to the object?**

**They are tangent to the object.**

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**How do the streaks relate to the data set and object?**

What are their slopes?

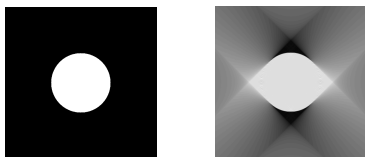
Either  $-1$  or  $+1$ —they are lines at the ends of the data set.

How do the streak lines relate to the object?

They are tangent to the object.

**New Moral:** Lines at the ends of the data set (min. or max. slope) that are tangent to the object can cause streak artifacts in limited angle reconstructions.





- ▶ A boundary of the object is *visible* in the reconstruction if: it is *tangent* to a line in the data set!



- ▶ A boundary of the object is *visible* in the reconstruction if: it is *tangent* to a line in the data set!
- ▶ A boundary of the object is *invisible* (not seen) in the reconstruction if:









# Summary

- ▶ If a boundary of the object is *tangent* to a line in the data set, then it will (should) be *visible in the reconstruction*.

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- ▶ If a boundary of the object is *tangent* to a line in the data set, then it will (should) be *visible in the reconstruction*.
- ▶ If a boundary of the object *is not tangent* to *any* line in the data set, then it will be *invisible* (or at least difficult to find) in the reconstruction.

# Summary

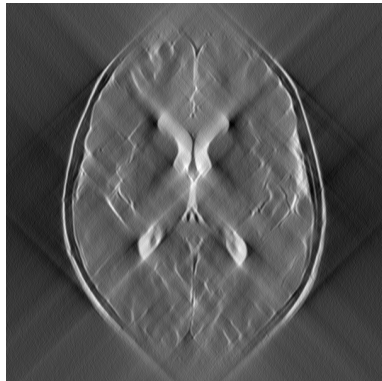
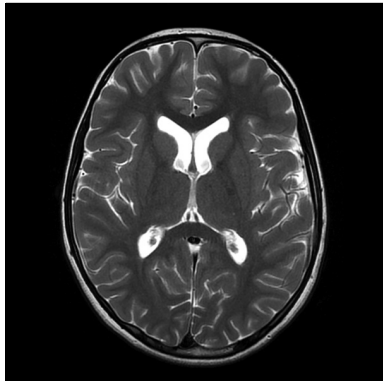
- ▶ If a boundary of the object is *tangent* to a line in the data set, then it will (should) be *visible in the reconstruction*.
- ▶ If a boundary of the object is *not tangent* to *any* line in the data set, then it will be *invisible* (or at least difficult to find) in the reconstruction.
- ▶ If a boundary of the object is *tangent* to a line *at an end of the data set*, then it will create a streak in the reconstruction along that line.

# Summary

- ▶ If a boundary of the object is *tangent* to a line in the data set, then it will (should) be *visible in the reconstruction*.
- ▶ If a boundary of the object is *not tangent* to *any* line in the data set, then it will be *invisible* (or at least difficult to find) in the reconstruction.
- ▶ If a boundary of the object is *tangent* to a line *at an end of the data set*, then it will create a streak in the reconstruction along that line.
- ▶ Where is the math?
  - ▶ Geometry was used to describe visible and invisible boundaries + streaks.
  - ▶ These observations are justified using deep mathematics (microlocal analysis) related to calculus.

## Example

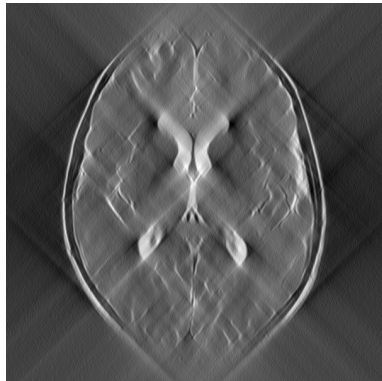
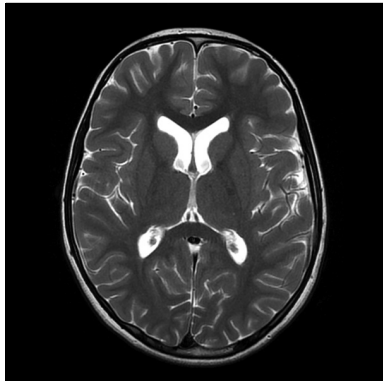
Limited angle CT data of a brain. **What lines are in the data set?**



Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

## Example

Limited angle CT data of a brain. **What lines are in the data set?**



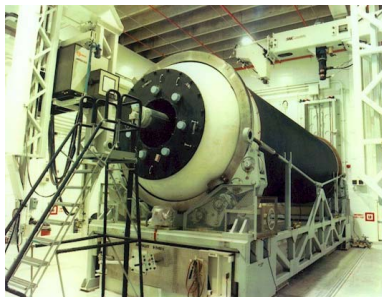
Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

- *Which features of the brain are visible in the reconstruction? Which are invisible?*
- *Where are the added streak artifacts?*



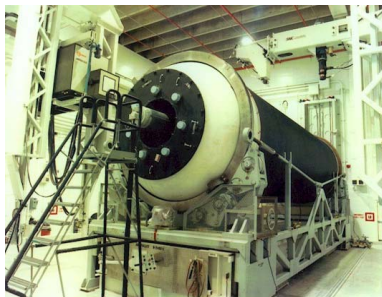
# Exterior Tomography

**Exterior Tomography:** tomography using only lines that are outside a disk to reconstruct the region outside the disk.



# Exterior Tomography

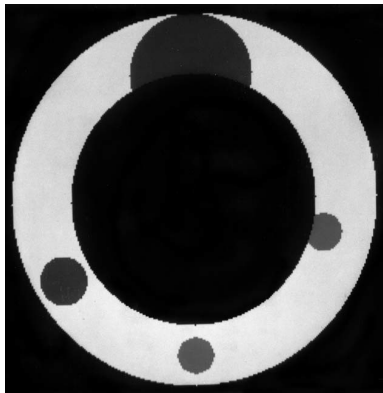
**Exterior Tomography:** tomography using only lines that are outside a disk to reconstruct the region outside the disk.



Exterior CT is used to evaluate of rockets because industrial X-ray CT scanners can't penetrate the thick central part of the rocket, but they can penetrate the outside annulus. Often scientists are interested in cracks, etc., in the rocket shell, anyway.

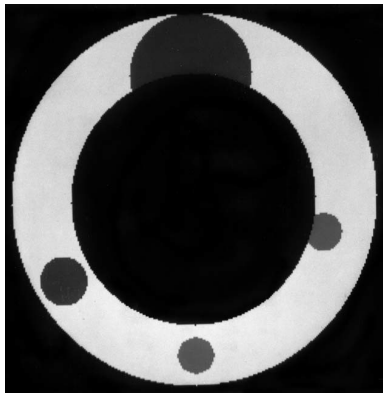
## Example

What boundaries will be easy to see in an exterior reconstruction of the phantom on the left?



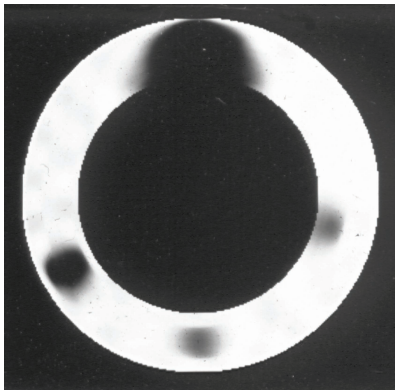
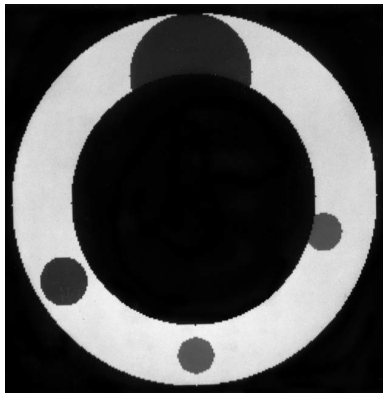
## Example

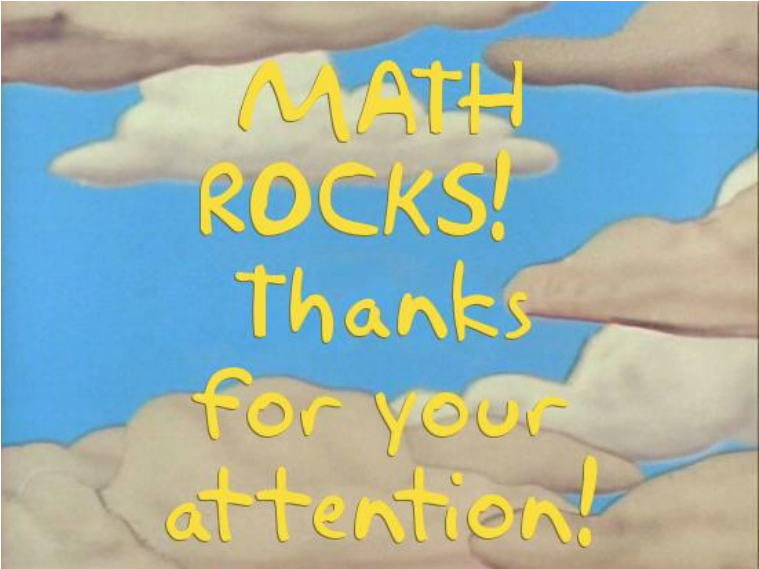
What boundaries will be easy to see in an exterior reconstruction of the phantom on the left?



## Example

What boundaries will be easy to see in an exterior reconstruction of the phantom on the left? [Q1988]









MATH  
ROCKS!  
Thanks  
for your  
attention!

# For Further Reading I


## *General references:*

-  Frank Natterer, *The Mathematics of Computerized Tomography*, Wiley, New York, 1986 (SIAM 2001).
-  Frank Natterer, Frank Wuebbeling, *Mathematical Methods in Image Reconstruction*, SIAM, 2001.




## *Introductory*

-  Peter Kuchment, The Radon transform and medical imaging. CBMS-NSF Regional Conference Series in Applied Mathematics, 85. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014. xvi+240 pp.
-  Gestur Olafsson, E.T. Quinto, The Radon Transform, Inverse Problems, and Tomography, (Proceedings of the 2005 AMS Short Course, Atlanta, GA) Proceedings of Symposia in Applied Mathematics, vol. 63, 2006.

## For Further Reading II




-  E.T. Quinto, An Introduction to X-ray tomography and Radon Transforms, Proceedings of Symposia in Applied Mathematics, Vol. 63, 2006, pp. 1-24.

### *Local and Lambda CT*



-  A. Faridani, E.L. Ritman, and K.T. Smith, *SIAM J. Appl. Math.* **52**(1992), 459–484,  
+Finch II: **57**(1997) 1095–1127.
-  A. Katsevich, Cone Beam Local Tomography, *SIAM J. Appl. Math.* 1999, Improved: Inverse Problems 2006.
-  A. Louis and P. Maaß, *IEEE Trans. Medical Imaging*, **12**(1993), 764-769.





## *Microlocal references:*



-  ***Intro + Microlocal:*** Microlocal Analysis in Tomography, joint with Venkateswaran Krishnan, chapter in Handbook of Mathematical Methods in Imaging, 2e, pp. 847-902, Editor Otmar Scherzer, Springer Verlag, New York, 2015  
[www.springer.com/978-1-4939-0789-2](http://www.springer.com/978-1-4939-0789-2)
-  Petersen, Bent E., Introduction to the Fourier transform & pseudodifferential operators. Monographs and Studies in Mathematics, 19. Pitman (Advanced Publishing Program), Boston, MA, 1983. xi+356 pp. ISBN: 0-273-08600-6
-  Strichartz, Robert, A guide to distribution theory and Fourier transforms. Reprint of the 1994 original [CRC, Boca Raton; MR1276724]. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. x+226 pp. ISBN: 981-238-430-8

# For Further Reading IV

-  Taylor, Michael Pseudo differential operators. Lecture Notes in Mathematics, Vol. 416. Springer-Verlag, Berlin-New York, 1974. iv+155 pp.
-  Taylor, Michael E. Pseudodifferential operators. Princeton Mathematical Series, 34. Princeton University Press, Princeton, N.J., 1981. xi+452 pp. ISBN: 0-691-08282-0

## *References to the work in the talk:*

-  E.T. Quinto, *SIAM J. Math. Anal.* **24**(1993), 1215-1225.
-  Characterization and reduction of artifacts in limited angle tomography, joint with Jürgen Frikel, *Inverse Problems*, 29 (2013) 125007 (21 pages). See also <http://iopscience.iop.org/0266-5611/labtalk-article/55769>

-  Artifacts in incomplete data tomography with applications to photoacoustic tomography and sonar, joint with Jürgen Friel, SIAM J. Appl. Math., 75(2),(2015) 703-725. (23 pages) Preprint on arXiv: <http://arxiv.org/abs/1407.3453>.
-  A paradigm for the characterization of artifacts in tomography, joint with Jürgen Friel. (arXiv: <http://arxiv.org/abs/1409.4103>). (an application of the algorithm in [ibid.] to an instructive but simpler case).